« Quadratic » Hawkes processes
(for financial price series)

Fat-tails and Time Reversal Asymmetry

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(building on previous work with
Rémy Chicheportiche & Steve Hardiman)

« Stylized facts »

I. Well known:

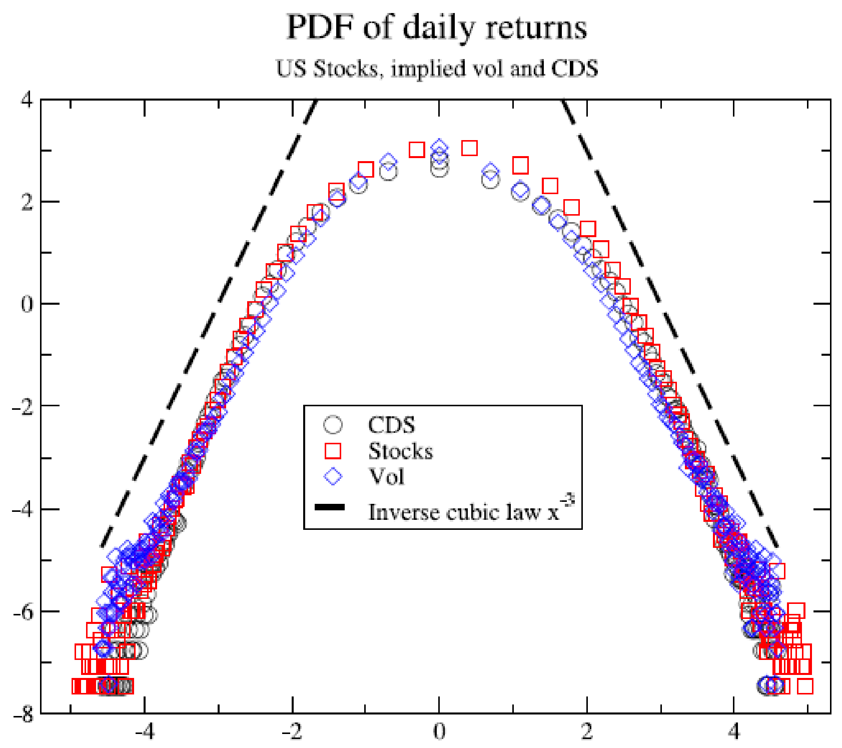
- Fat-tails in return distribution

$$p(r) \underset{|r|\to\infty}{\sim} \frac{C'}{|r|^{1+\nu}}$$

with a (universal?) exponent ν around 4 for many different assets, periods, geographical zones,…

- Fluctuating volatility with « long-memory »

- Leverage effect (negative return/vol correlations)



With Ch. Biely, J. Bonart

« Stylized facts »

II. Less well known:

- Time Reversal Asymmetry (TRA) in realized volatilities:

$$\langle r_t^2 \sigma_{t+\tau}^2 \rangle_t > \langle r_{t+\tau}^2 \sigma_t^2 \rangle_t .$$

Past large-scale vol. (r^2) better predictor of future realized (HF) vol. than vice-versa: *The « Zumbach » effect*

- Intuition: past trends, up or down, increase future vol more than alternating returns (for a fixed HF activity)
- Reverse not true (HF vol does not predict more trends)

A bevy of models

$$r_t = \sigma_t \xi_t$$

- Stochastic volatility models (with Gaussian residuals)
  → Heston: no fat tails, no long-memory, no TRA
  → « Rough » fBM for log-vol with a small Hurst exponent H*: tails still too thin, no TRA

- GARCH-like models (with Gaussian residuals)
  → GARCH: exponentially decaying vol corr., strong TRA
  → FI-GARCH: tails too thin, TRA too strong

- None of these models are « micro-founded » anyway

(* Bacry-Muzy: H=0; Gatheral, Jaisson, Rosenbaum: H=0.1)

Hawkes processes

- A *self-reflexive feedback* framework, mid-way between purely stochastic and agent-based models
- Activity is a Poisson Process with history dependent rate:

$$\lambda_t = \lambda_\infty + \int_{-\infty}^{t} \phi(t-s)\ \mathrm{d}N_s$$

- Feedback intensity $n \equiv \int_0^\infty \phi(\tau) d\tau$ < 1
- Calibration on financial data suggests *near criticality* (n → 1) and *long-memory* power-law kernel ϕ : the « Hawkes without ancestors » limit (Brémaud-Massoulié)

Continuous time limit of near-critical Hawkes

- Jaisson-Rosenbaum show that when $n \rightarrow 1$ Hawkes processes converge (in the right scaling regime) to either:

  i) Heston for short-range kernels
  ii) Fractional Heston for long-range kernels, with a small
Hurst exponent H

- Cool result, but: still no fat-tails and no TRA…
- J-R suggest results apply to log-vol, but why?
- Calibrated Hawkes processes generate very little TRA, even on short time scales (see below)

Generalized Hawkes processes

- Intuition: not just past activity, but *price moves themselves* feedback onto current level of activity
- The most general quadratic feedback encoding is:

$$\lambda_t = \lambda_\infty + \frac{1}{\psi}\int_{-\infty}^{t} L(t-s)\,\mathrm{d}P_s + \frac{1}{\psi^2}\int_{-\infty}^{t}\int_{-\infty}^{t} K(t-s,t-u)\,\mathrm{d}P_s\,\mathrm{d}P_u$$

- With: $dN_t := \lambda_t\, dt$; $dP := (+/-)\, \psi\, dN$ with random signs

- L(.): leverage effect neglected here (small for intraday time scales)
- K(.,.) is a symmetric, positive definite operator
- Note: $K(t,t)=\phi(t)$ is exactly the Hawkes feedback ($dP^2=dN$)

Generalized Hawkes processes

$$\lambda_t = \lambda_\infty + \frac{1}{\psi}\int_{-\infty}^{t} L(t-s)\,\mathrm{d}P_s + \frac{1}{\psi^2}\int_{-\infty}^{t}\int_{-\infty}^{t} K(t-s,t-u)\,\mathrm{d}P_s\,\mathrm{d}P_u$$

- 1st order necessary condition for stationarity (for L(.)=0):

$$\overline{\lambda} = \frac{\lambda_\infty}{1-\mathrm{Tr}(K)} \rightarrow \begin{array}{c} \lambda_\infty > 0 \text{ and } \mathrm{Tr}(K) < 1 \\ \text{or } \lambda_\infty = 0 \text{ and } \mathrm{Tr}(K) = 1. \end{array}$$

Generalized Hawkes processes

$$\lambda_t = \lambda_\infty \; + \; \frac{1}{\psi}\int_{-\infty}^{t} L(t-s)\,\mathrm{d}P_s \; + \; \frac{1}{\psi^2}\int_{-\infty}^{t}\int_{-\infty}^{t} K(t-s,t-u)\,\mathrm{d}P_s\,\mathrm{d}P_u$$

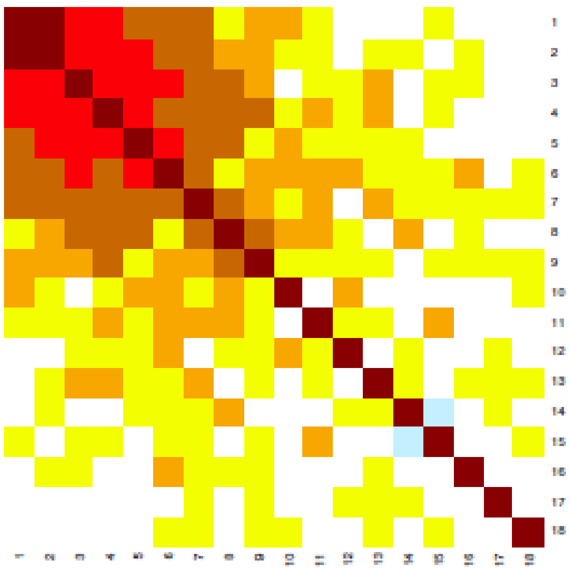
- 2- and 3-points correlation functions
- $$\mathcal{C}(\tau) \equiv \mathbb{E}\left[\frac{\mathrm{d}N_t}{\mathrm{d}t}\frac{\mathrm{d}N_{t-\tau}}{\mathrm{d}t}\right] - \overline{\lambda}^2 \;=\; \mathbb{E}\left[\lambda_t \frac{\mathrm{d}N_{t-\tau}}{\mathrm{d}t}\right] - \overline{\lambda}^2,$$
  $$\mathcal{D}(\tau_1,\tau_2) \equiv \frac{1}{\psi^2}\,\mathbb{E}\left[\frac{\mathrm{d}N_t}{\mathrm{d}t}\frac{\mathrm{d}P_{t-\tau_1}}{\mathrm{d}t}\frac{\mathrm{d}P_{t-\tau_2}}{\mathrm{d}t}\right] \;=\; \frac{1}{\psi^2}\,\mathbb{E}\left[\lambda_t \frac{\mathrm{d}P_{t-\tau_1}}{\mathrm{d}t}\frac{\mathrm{d}P_{t-\tau_2}}{\mathrm{d}t}\right]$$

$$\mathcal{C}(\tau) = \kappa\overline{\lambda}K(\tau,\tau) + \int_{-\infty}^{\tau} \mathrm{d}u\,K(\tau-u,\tau-u)\mathcal{C}(u) + 2\int_{0^+}^{\infty}\mathrm{d}u\int_{u^+}^{\infty}\mathrm{d}r\,K(\tau+u,\tau+r)\mathcal{D}(u,r),$$

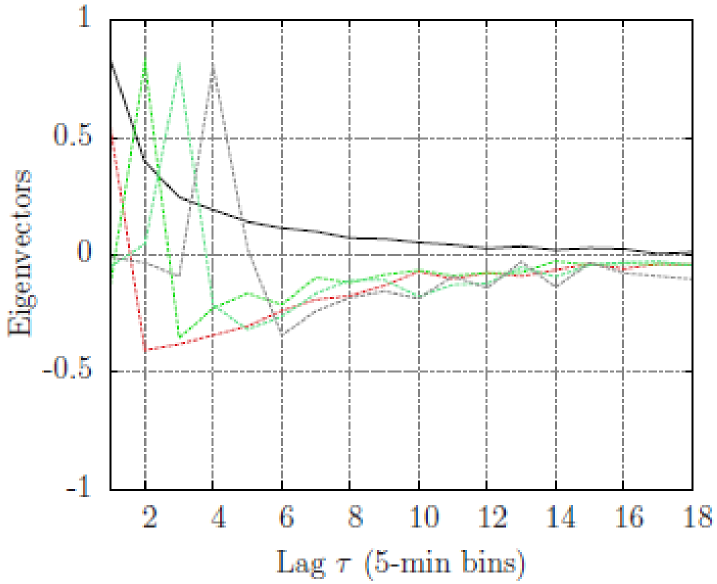
- And a similar <u>*closed*</u> equation for $\mathcal{D}(.,.)$, $\mathcal{C}(.)$
- This allows one to do a GMM calibration

Calibration on 5 minutes US stock returns

- Using GMM as a starting point for MLE, we get for K(s,t):



→



- K is well approximated by Diag + Rank 1:

$$K(\tau,\tau') \approx \phi(\tau)\delta_{\tau-\tau'} + k(\tau)k(\tau')$$

Calibration on 5 minutes US stock returns

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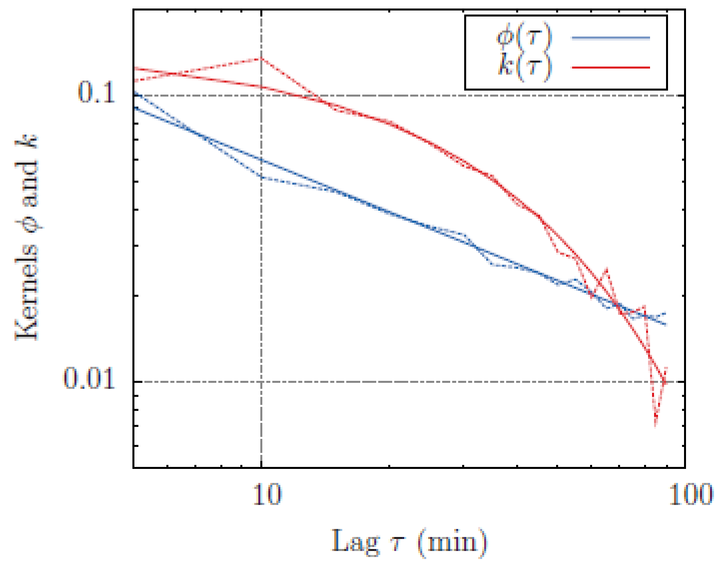


$$\phi(\tau) = g\tau^{-\alpha} \quad , \quad k(\tau) = k_0 \exp(-\omega\tau),$$

$$g = 0.09,\ \alpha = 0.60,\ k_0 = 0.14,\ \omega = 0.15$$

→ Tr(K) (intraday) = 0.74 (Diag) + 0.06 (Rank 1) = 0.8

Generalized Hawkes processes: Hawkes + « ZHawkes »

$$K(\tau,\tau') \approx \phi(\tau)\delta_{\tau-\tau'} + k(\tau)k(\tau')$$

$$\lambda_t = \lambda_\infty + H_t + Z_t^2,$$

$$H_t := \int_{-\infty}^{t} \phi(t-s)\,\mathrm{d}N_s, \qquad Z_t = \frac{1}{\psi}\int_{-\infty}^{t} k(t-s)\,\mathrm{d}P_s.$$

Z_t : moving average of price returns, i.e. recent « trends »

→ The Zumbach effect: trends increase future volatilities

The Markovian Hawkes + ZHawkes processes

$$\lambda_t = \lambda_\infty + H_t + Z_t^2,$$

$$H_t := \int_{-\infty}^{t} \phi(t-s)\, \mathrm{d}N_s, \qquad Z_t = \frac{1}{\psi} \int_{-\infty}^{t} k(t-s)\, \mathrm{d}P_s.$$

With: $k(t) = \sqrt{2 n_Z \omega}\, \exp(-\omega t)$ and $\phi(t) = n_H \beta \exp(-\beta t)$

<u>In the continuum time limit</u>: ($h = H$; $y = Z^2$):

$dh = [-(1-n_H)\, h + n_H\, (\lambda + y)\,]\, \beta\, dt$

$dy = [-(1-n_Z)\, y + n_Z\, (\lambda + h)\,]\, \omega\, dt + [2\, \omega\, n_Z\, y\, (\lambda + y + h)]^{1/2}\, dW$

→ 2-dimensional generalisation of Pearson diffusions ($n_H = 0$)

The Markovian Hawkes + ZHawkes processes

$$dh = [-(1-n_H)\, h + n_H\, (\lambda + y)\,]\, \beta\, dt$$

$$dy = [-(1-n_Z)\, y + n_Z\, (\lambda + h)\,]\, \omega\, dt + [2\, \omega\, n_Z\, y\, (\lambda + y + h)]^{1/2}\, dW$$

- For large y: $P_{st.}(h|y) = 1/y\, F(h/y)$ (i.e h is of order y)

→ The y process is asymptotically multiplicative, as assumed in many « log-vol » models (including Rough vols.)

→ One can establish a 3rd order ODE for the L.T. of F(.)
→ This can be explicitely solved in the limits

$$\beta >> \omega \text{ or } \omega >> \beta \text{ or } n_Z \to 0 \text{ or } n_H \to 0$$

The Markovian Hawkes + ZHawkes processes

$$dh = [- (1-n_H)\, h + n_H\, (\lambda + y)\,]\, \beta\, dt$$

$$dy = [- (1-n_Z)\, y + n_Z\, (\lambda + h)\,]\, \omega\, dt + [2\, \omega\, n_Z\, y\, (\lambda + y + h)]^{1/2}\, dW$$

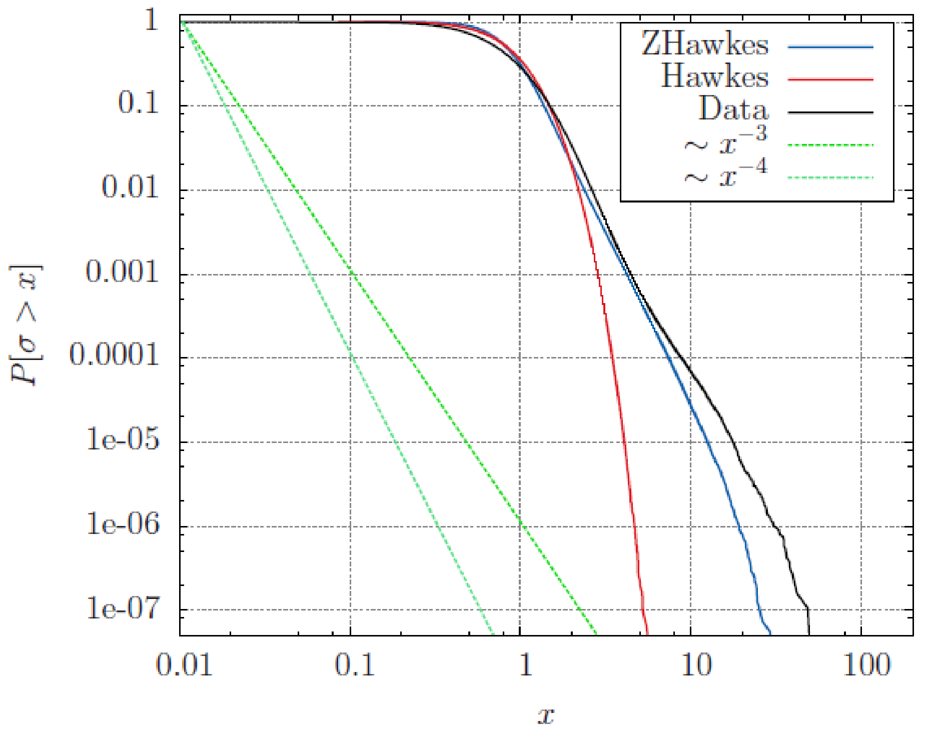
→ The upshot is that the vol/return distribution has a power-law tail with a computable exponent, for example:

* $\beta >> \omega$ → $\nu = 1 + (1 - n_H)/n_Z$

* $n_Z \to 0$ → $\nu = 1 + b(\omega/\beta,\, n_H)/n_Z$

→ Even when n_Z is smallish, n_H conspires to drive the tail exponent ν in the empirical range ! – see next slide

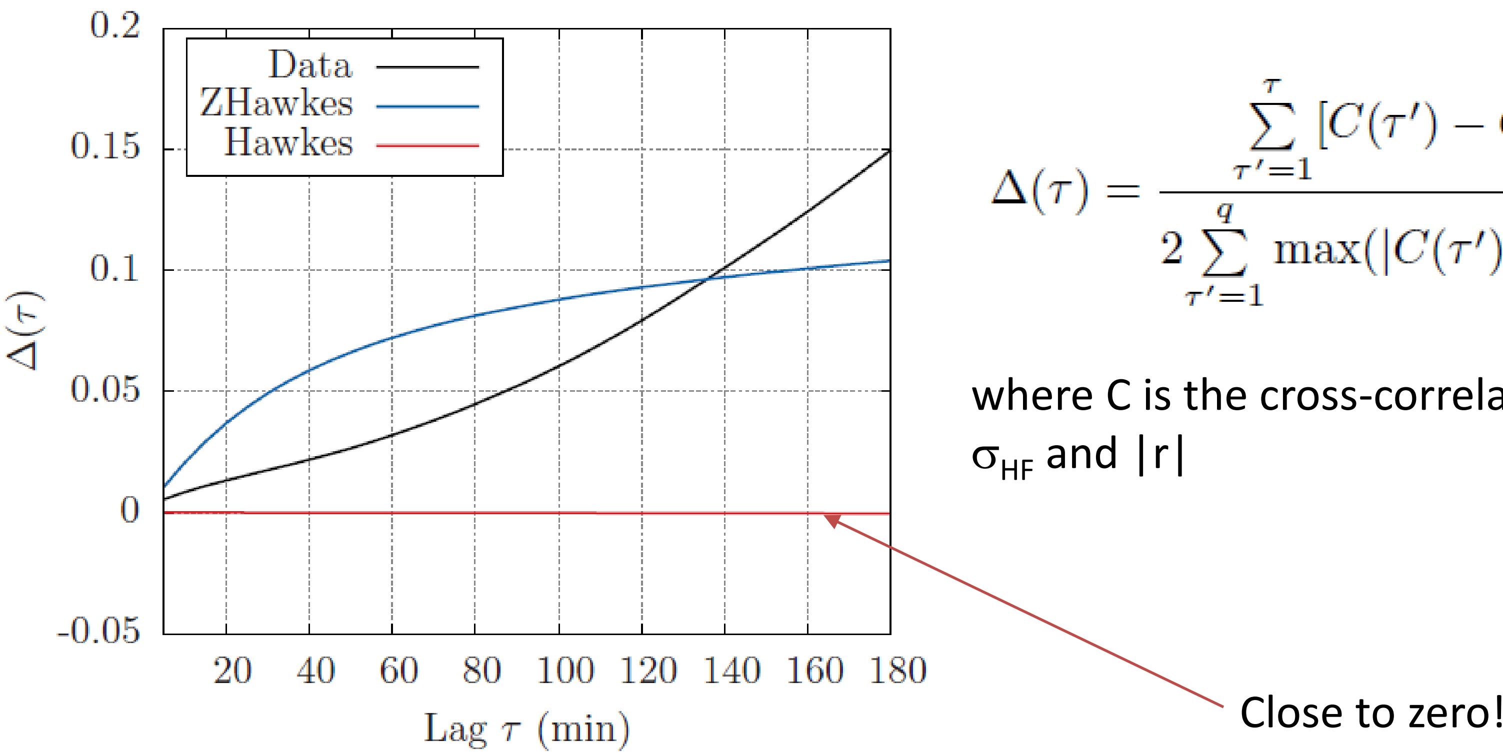
The <u>calibrated</u> Hawkes + ZHawkes process: numerical simulations



Fat-tails are indeed accounted for with $n_Z = 0.06$!

Note: $\Delta P_\tau = \pm\psi$ so tails ***do not*** come from residuals

The calibrated Hawkes + ZHawkes process: numerical simulations



$$\Delta(\tau) = \frac{\sum_{\tau'=1}^{\tau} [C(\tau') - C(-\tau')]}{2\sum_{\tau'=1}^{q} \max(|C(\tau')|, |C(-\tau')|)}$$

where C is the cross-correlation between σ_{HF} and $|r|$

The level of TRA is also satisfactorily reproduced

(wrong concavity probably due to intraday non-stationarities not accounted for here)

Conclusion

- Generalized Hawkes Processes: a natural extension of Hawkes processes accounting for « trend » (Zumbach) effects on volatility – a step to close the gap between ABMs and stochastic models
- Leads naturally to a multiplicative « Pearson » type (2d) diffusion for volatility
- Accounts for tails (induced by micro-trends) and TRA
- GHP can have long memory without being critical

- A lot of work remaining (empirical and mathematical)
- Non-stationarity + Extension to daily time scales (O/I)??
- Real « Micro » foundation ? Higher order terms ?